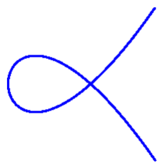
TOPOLOGY - III, EXERCISE SHEET 8

Exercise 1. *Local homology.*

Let X be a topological space and let $x \in X$ be a closed point. Recall that we define the local homology groups of X at x to be the groups $H_k(X, X - \{x\})$ for all k.

(1) Show that $H_k(X, X - \{x\}) \cong H_k(U, U - \{x\})$ for any open neighbourhood U of x in X.
(2) Let M be a topological manifold of dimension n. That is, the space M is Hausdorff and every point in M has an open neighbourhood homeomorphic to $\mathbb{R}^n$. For all $x \in M$ and $k \geq 0$,compute the local homology groups $H_k(M, M - \{x\})$.
(3) Let G be a graph. That is, a collection of vertices and edges. Then observe that G can be thought of as a $\Delta-$ complex of dimension 1. Let v be a vertex of G, compute the local homology groups $H_k(G, G - \{v\})$ for all k in terms of the degree of the vertex v.
(4) Let $C \subseteq \mathbb{R}^2$ denote the curve cut-out by the equation $y^2 - x^3 - x = 0$. Using local homology, show that C is not a topological manifold.
Hint:



Exercise 2. *Equivalence of simplicial and singular homology.*

Let X be a $\Delta-$complex and let X^k denote the $k-$skeleton of X. That is X^k is the union of all $i-$ simplices in X, with $i \leq k$. It was shown in the lectures that the map $\Delta_n(X) \to C_n(X)$ which sends an open $n-$simplex e^n_α to its characteristic map $\sigma^n_\alpha : \Delta^n_\alpha \to X_k$, induces an isomorphism $H^\Delta_n(X^k) \xrightarrow{\sim} H_n(X^k)$ for all k and n. In particular if X is finite dimensional, this shows the equivalence of simplicial and singular homology for X since $X = X^k$ for some k. The goal of this exercise is to give isomorphisms $H^\Delta_n(X) \xrightarrow{\sim} H_n(X)$ in the general case as well.

(1) Let C be a compact subset of a $\Delta-$complex X. Show that C meets only finitely many open simplices of X.
Hint: If not, try to construct an open cover of C with no finite sub-cover.
(2) Using part (1) and the result proved in the lecture conclude that $H^\Delta_n(X) \cong H_n(X)$ for all n.
Hint: Try to prove surjectivity and injectivity individually and observe that (1) makes both these questions boil down to the ones for the finite dimensional case.

(3) Now let A be a sub-complex of X. That is, the subspace A is a union of closed simplices of X. Show that $H_n^{\Delta}(X, A) \cong H_n(X, A)$ for all n.
Hint: Use (2) and the five-lemma.